On the Parameterized Complexity of the Structure of Lineal Topologies (Depth-First Spanning Trees) of Finite Graphs: The Number of Leaves

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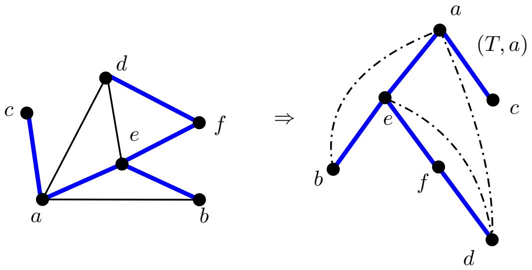

A graph G and a DFS tree (T, a) of G with back edges

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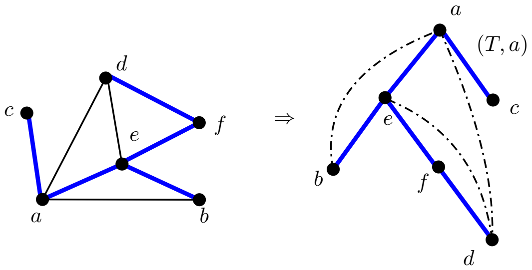

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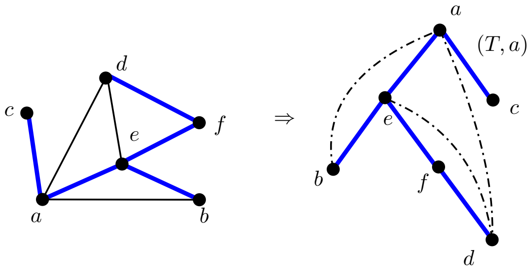

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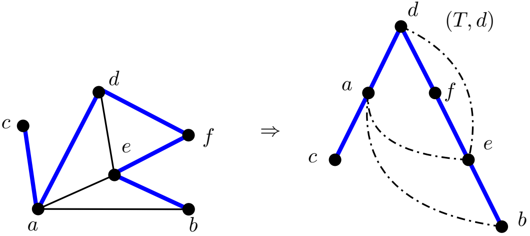

The spanning tree (T, d) is not a DFS tree of G.

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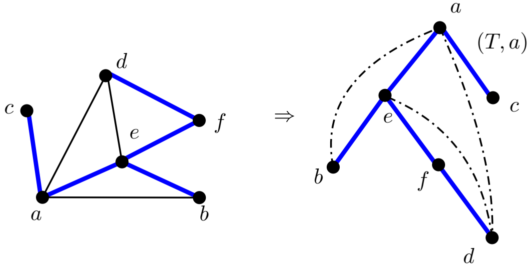

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- This notion corresponds to a point-set topology on $E(G)$ defined by:

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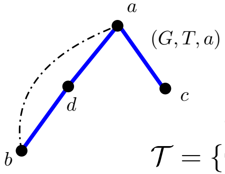
- For example, given :



a (T, a)

$$\mathcal{T} = \{\emptyset, \{ad\}, \{ac\}, \{ad, db, ba\}, \{ad, ac\}, \{ad, db, ba, ac\}\}.$$

A topology on $E(G)$

Applications

- Planarity testing and embedding [De Fraysseix, 2008; Hopcroft and Tarjan, 1974]

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- DFS trees have been used to structure the search space of backtracking algorithms for solving *constraint satisfaction problems* [Freuder and Quinn, 1985]
- The problem k-Min-HLT of asking, for a given graph G and a positive integer k, whether G has an (LT with height $h \leqslant k$) is NP-complete. [Fellows et al., 1988]

Problem Definitions

k-Minimum Leafy Lineal Topology (k-Min-LLT)

Input: A connected undirected graph $G = (V, E)$ and $k \in \mathbb{N}$
Question: Does G admit an LT with $\leqslant k$ leaves?

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- k-Min-LLT is NP-complete.

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Dual Min LLT

Input:	A connected undirected graph $G = (V, E)$ and $k \in \mathbb{N}$
Parameter:	k
Question:	Does G admit an LT with $\leqslant n - k$ leaves?

Dual Max LLT

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The k-Max-LLT *problem is W[1]-hard parameterized by* k

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- Dual Min-LLT and Dual Max-LLT are NP-complete when considered classically.

Theorem

Dual Min-LLT *and* Dual Max-LLT *are FPT parameterized by* k*.*

Proof:

We present a parameterized reduction from:

Multicolored Independent Set (MIS)

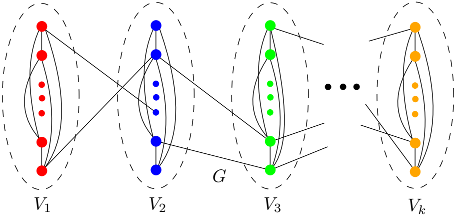
Input:	A graph $G = (V, E)$, and $f : V \to [1, k]$ with $k \in \mathbb{N}$.
Parameter:	k
Question:	Does G contain a k-colored independent set?

We assume that each color class V_i, for $i \in [1, k]$, induces a clique.

W[1]-Hardness of k-Max-LLT Parameterized by k

Proof:

- Given an instance (G, k) of MIS.



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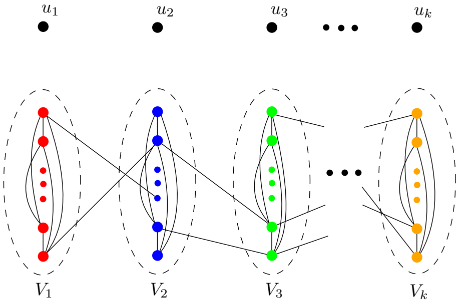

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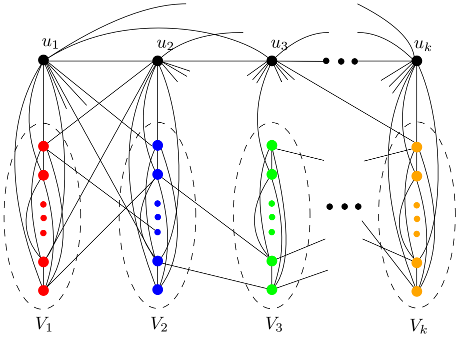

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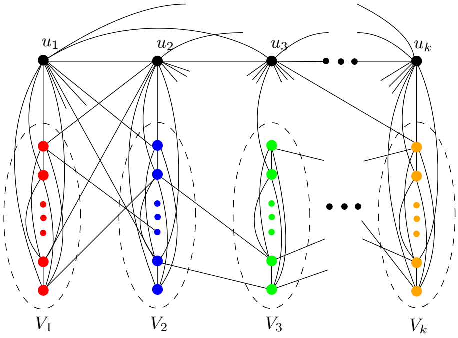
- (G', k') can be constructed in polynomial time.



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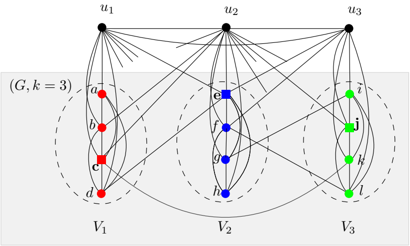
Lemma

G has a k-colored independent set $\Rightarrow$ G' admits an LT with at least k leaves.

Proof.

- Let $X = \{x_1, ..., x_k\}$ be a k-colored independent set in G.



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- Let $X = \{x_1, ..., x_k\}$ be a k-colored independent set in G.
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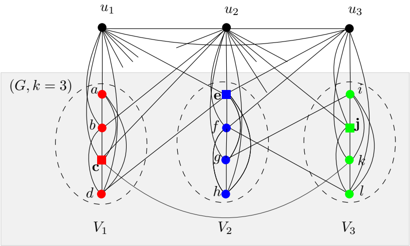

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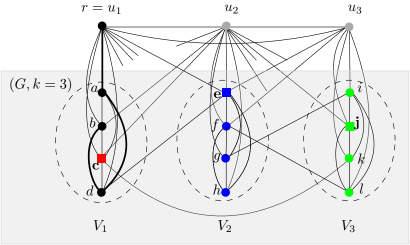

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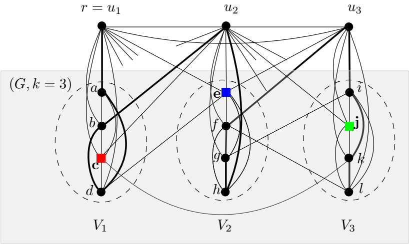

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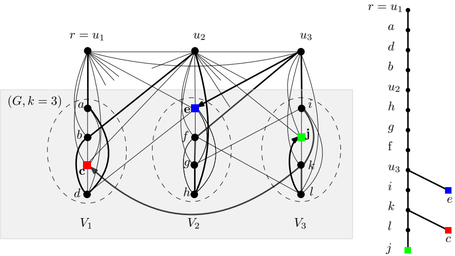

G' with $X = \{c, e, j\}$, and the resulting DFS tree T

Lemma

G has a k-colored independent set $\Leftarrow$ G' admits an LT with at least k leaves.

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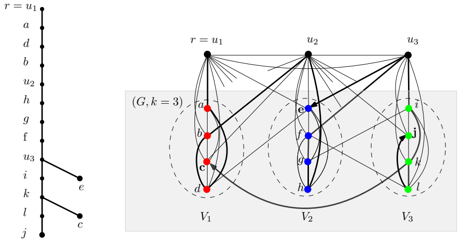
- Suppose that $k \geqslant 2$, and G' admits a DFS tree T' with at least k leaves.



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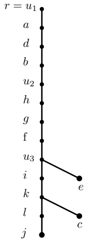

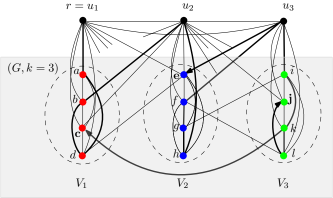

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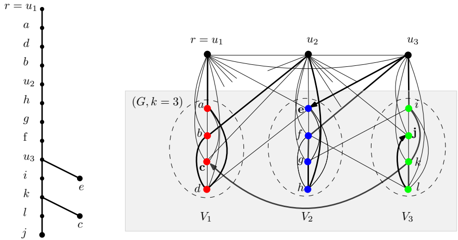
- *Claim 1: Each vertex in X belongs to at most one color class V_i in G.*



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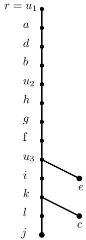
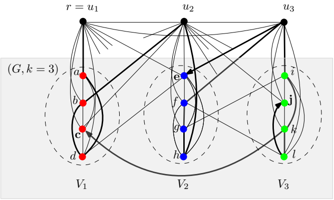
- *Claim 1: Each vertex in X belongs to at most one color class V_i in G.*
- *Claim 2: None of the vertices in X belongs to $U = \{u_1, \ldots, u_k\}$.*





FPT Algorithms

Dual Max-LLT and Dual Min-LLT parameterized by k

Method

We employ the following theorem:

Theorem (Courcelle, 1990)

Given $k \in \mathbb{N}$ and a fixed monadic second-order logic (MSO) formula ϕ of length ℓ expressing a graph property, there is an algorithm that takes G with treewidth at most k as input and decides whether $G \models \phi$ in time $\mathcal{O}\big(f(\ell, k) \cdot n\big)$, for some computable function f.

We express in MSO_1 (a version of MSO) the property of having:

- an LT with at most $n - k$ leaves
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Plan:

Given a DFS tree T resulting from any DFS of the graph G, T either:

- solves the problem trivially, or
- yields a bounded path decomposition.

FPT Algorithm for Dual Max-LLT

Observation

For any given graph G and $k \in \mathbb{N}$, if G is a YES-instance of Dual Max-LLT, *then G admits an LT with height at most k.*

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Algorithm

Let T be a DFS tree of G given by DFS.

- If the height h of T is more than $2^{k+1} - 2$, then return NO.

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FPT Algorithm for Dual Min-LLT

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Difference:

- If the number of leaves of T is at most $n - k$, then return YES.
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