Kernelization for Finding Lineal Topologies (Depth-First Spanning Trees) with Many or Few Leaves

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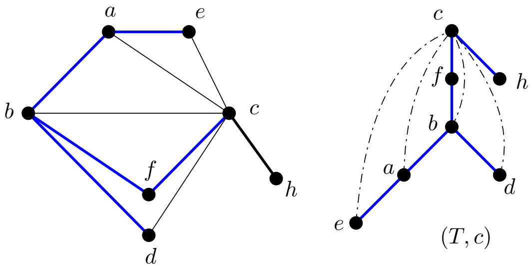
Introduction

- Given a connected undirected graph G, a *depth-first spanning* (DFS) tree is a rooted spanning tree T of G with the property that
  - for every edge $xy \in E(G) \setminus E(T)$, either x is a descendant of y with respect to T, or x is an ancestor of y.



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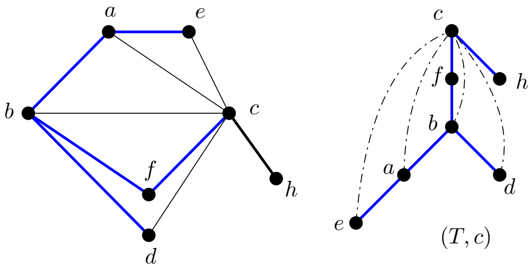
- The edges $E(G) \setminus E(T)$ are called *back edges*.



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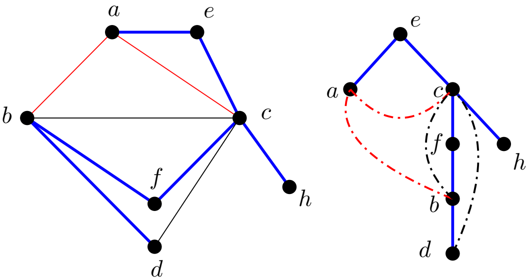

This spanning tree (T, e) is not a DFS tree of G.

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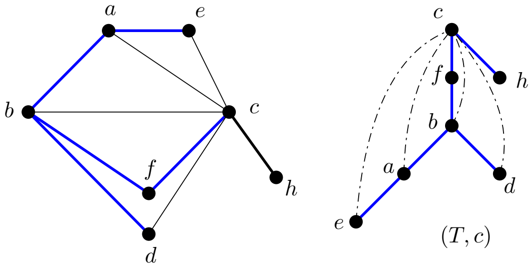

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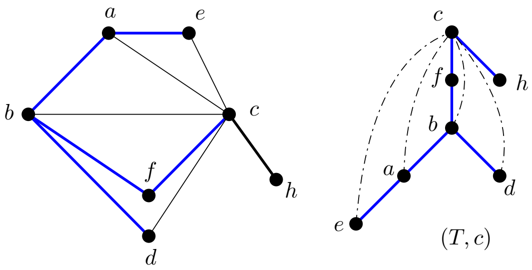

A graph G and a DFS tree (T, c) of G rooted at $c \in V(G)$

- A DFS tree is also called *lineal topology* (LT) [Sam et al., 2023]
- *Leafy lineal topology* (LLT): an LT with many or few leaves.

Computing a Leafy Lineal Topology (LLT)

[Sam et al., 2023]
Input: *A graph* $G = (V, E)$ *and* $k \in \mathbb{N}$*.*
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- Dual Min-LLT and Dual Max-LLT are **FPT** parameterized by k.

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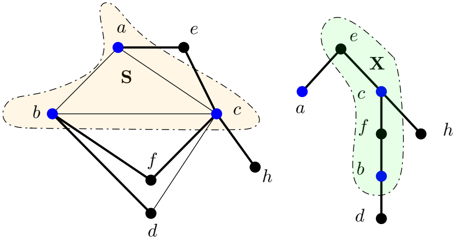
- Dual Min-LLT and Dual Max-LLT can be solved in $k^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ time.
- Min-LLT and Max-LLT parameterized by the vertex cover τ of G admit kernels with $\mathcal{O}(\tau^3)$ vertices.

Lemma (1)

Let S be a vertex cover of G of size s. Then, every rooted spanning tree T of G has at most $2s$ internal vertices, and at most s of these internal vertices are not in S.



An example

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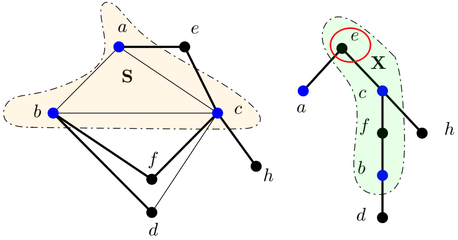

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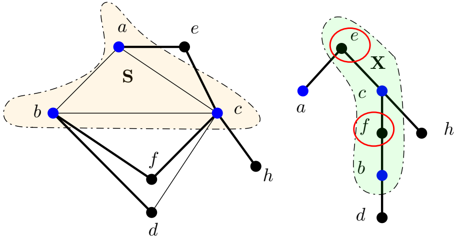

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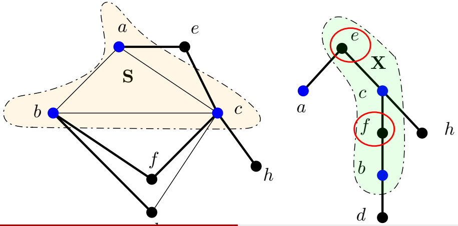

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- $|X \backslash S| \leqslant s$ and $|X| \leqslant 2s$.

Lemma (2)

There is a polynomial-time algorithm that, given a vertex cover S of G of size s, outputs a graph G' with at most $s^2(s-1)+3s$ vertices such that for every integer $t \geqslant 0$:

- *G has a DFS tree with exactly t internal vertices if and only if G' has a DFS tree with exactly t internal vertices.*

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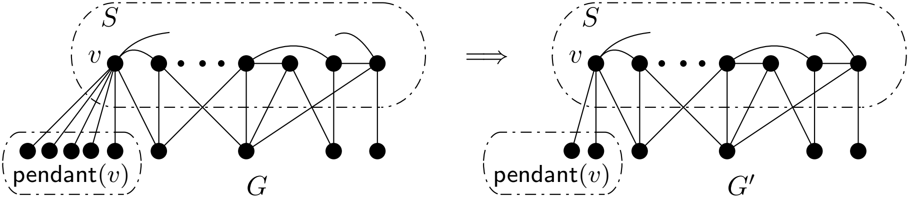
We apply two reduction rules.

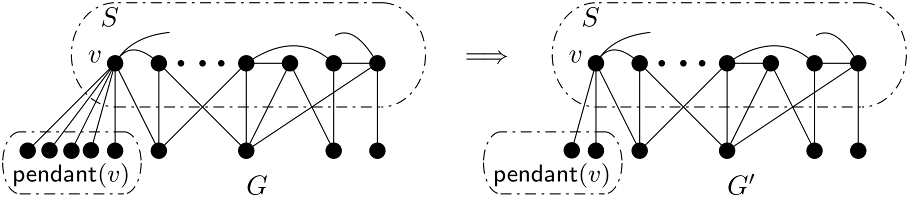
Rule 1



foreach $v \in S$ **do**
- **if** $|\text{pendant}(v)| > 2$ **then**
  - delete all but two vertices in pendant(v) from G
- **end**

end

Safeness of Rule 1



foreach $v \in S$ **do**
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    delete all but two vertices in pendant(v) from G
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Observation: For each $v \in S$, if $|\text{pendant}(v)| > 2$, then at most one vertex of pendant(v) is the root and the rest are leaves of any DFS tree of G.

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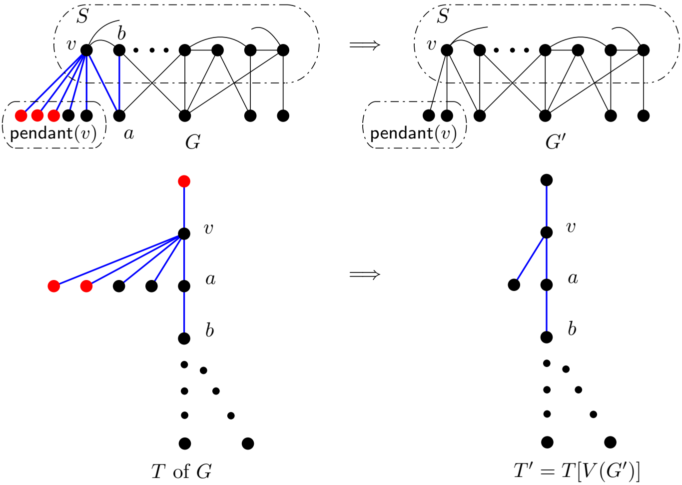

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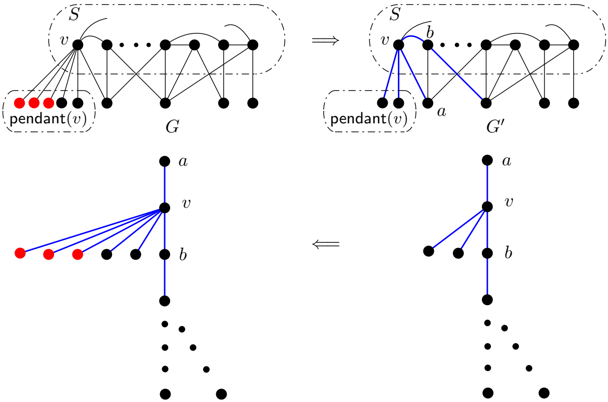

$T = T'[V(G') + V(G) \setminus V(G')]$

A DFS tree T' of G'

Rule 2

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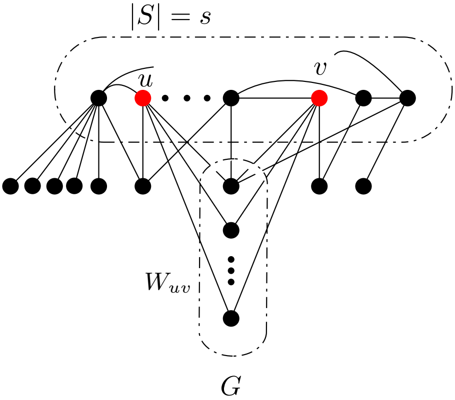

forall *pairs* $\{u, v\}$ *of distinct vertices of* S **do**

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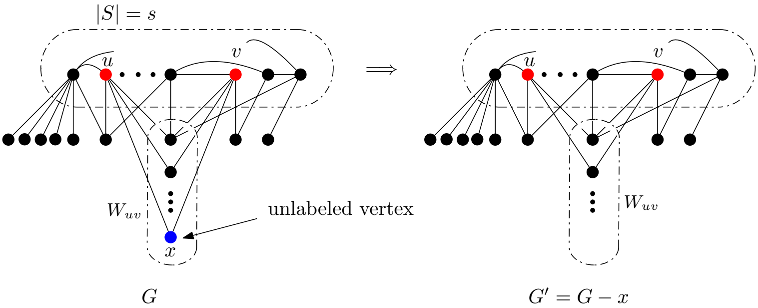

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Delete the unlabeled vertices of $V(G)\backslash S$ with at least two neighbors in S.

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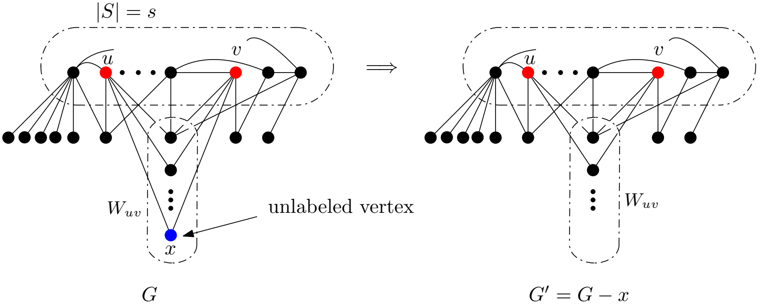

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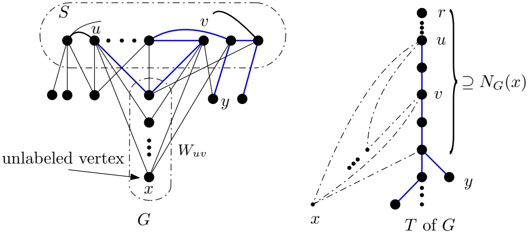
Delete the unlabeled vertices of $V(G) \backslash S$ with at least two neighbors in S.

Observation: *If* $|W_{uv}| \geqslant 2s + 1$ *then any* T *has at most* s *internal vertices and at least* $s + 1$ *leaves from* W_{uv}*.*

Claim 1:

(i) For any DFS tree T of G, the vertices of $N_G(x)$ are vertices of a root-to-leaf path of T.



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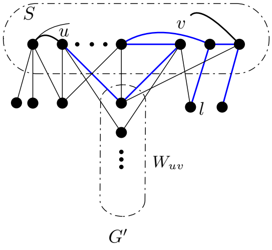

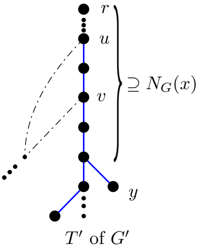

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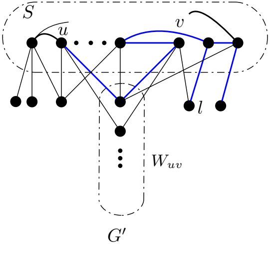

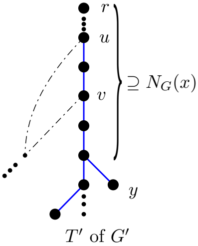

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We show that G has a DFS tree with exactly t internal vertices if and only if $G' = G - x$ has a DFS tree with exactly t internal vertices.

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- Thus, G' has at most $s^2(s-1) + 2s + s = s^2(s-1) + 3s$ vertices.

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We showed that

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  - **Open problem:** Do these problems have linear kernels?

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Thank you!