Kernelization for Finding Lineal Topologies (Depth-First Spanning Trees) with Many or Few Leaves

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Friday Seminar

Introduction

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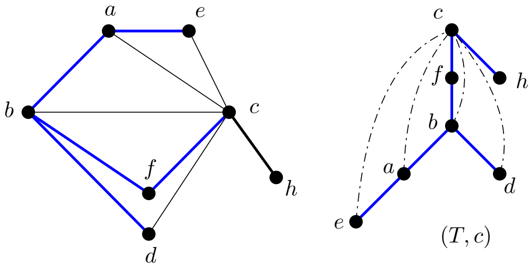
  - for every edge $xy \in E(G) \setminus E(T)$, either x is a descendant of y with respect to T, or x is an ancestor of y.



A graph G and a DFS tree (T, c) of G rooted at $c \in V(G)$

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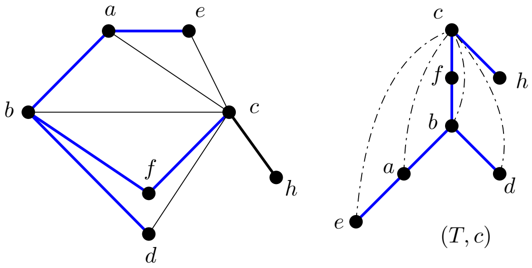
- The edges $E(G) \backslash E(T)$ are called *back edges*.



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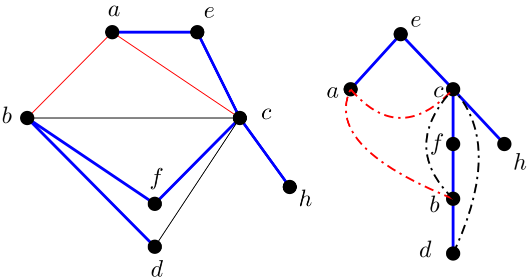

A spanning tree (T, e) of G that is not a DFS tree.

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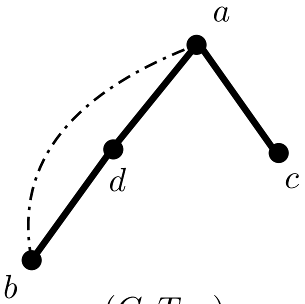

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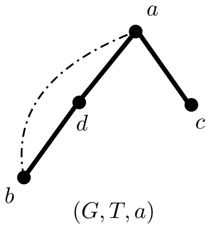

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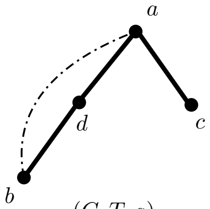

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- *Leafy lineal topology* (LLT): an LT with many or few leaves.

Computing a Leafy Lineal Topology (LLT)

[Sam et al., 2023]
Input: *A graph* $G = (V, E)$ *and* $k \in \mathbb{N}$.
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- Min-LLT is NP-hard and **Para-NP-hard** parameterized by k.
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- Dual Min-LLT and Dual Max-LLT are **FPT** parameterized by k.

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This work:

- Dual Min-LLT and Dual Max-LLT can be solved in $k^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ time.

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- Min-LLT and Max-LLT parameterized by the vertex cover τ of G admit kernels with $\mathcal{O}(\tau^3)$ vertices.

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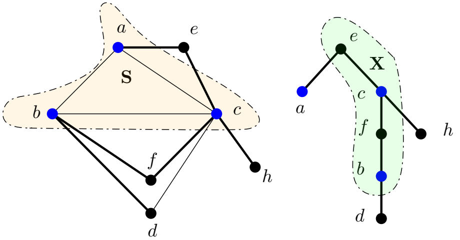
Let S be a vertex cover of G of size s. Then, every rooted spanning tree T of G has at most $2s$ internal vertices, and at most s of these internal vertices are not in S.



An example

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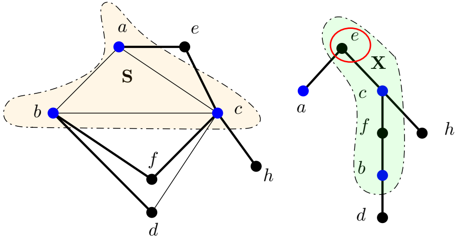

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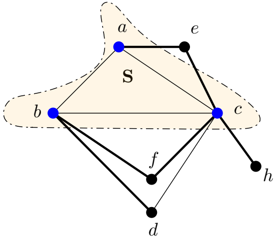

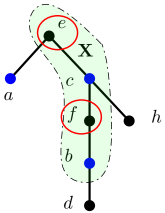

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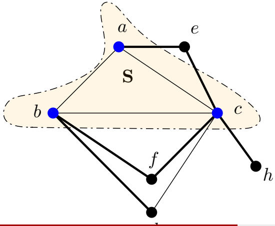

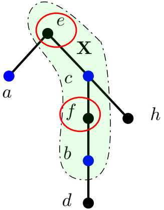

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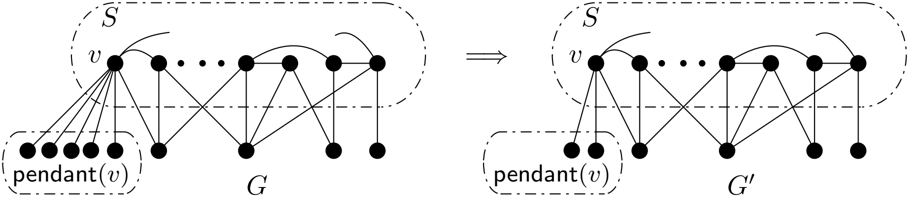
- $|X \backslash S| \leqslant s$ and $|X| \leqslant 2s$.

Lemma (2)

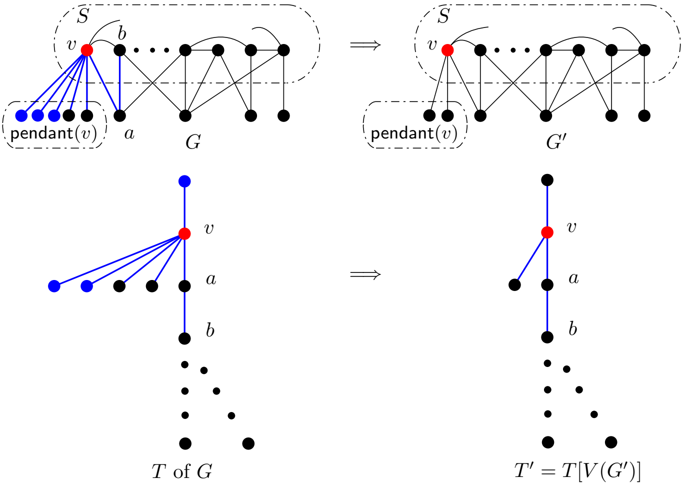
There is a polynomial-time algorithm that, given a vertex cover S of G of size s, outputs a graph G' with at most $s^2(s-1)+3s$ vertices such that for every integer $t \geqslant 0$:

- *G has a DFS tree with exactly t internal vertices if and only if G' has a DFS tree with exactly t internal vertices.*



foreach $v \in S$ **do**
    if $|\text{pendant}(v)| > 2$ **then**
        delete all but two vertices in pendant(v) from G
    end
end

Safeness of Rule 1



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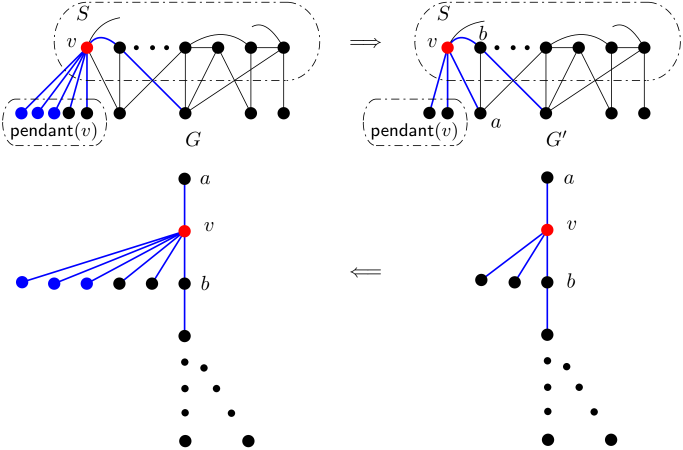

$T = T'[V(G') + V(G) \setminus V(G')]$

A DFS tree T' of G'

Rule 2

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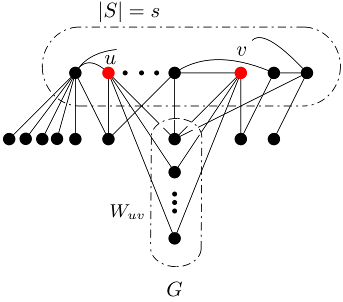

forall *pairs* $\{u, v\}$ *of distinct vertices of* S **do**

  if $|W_{uv}| > 2s$ **then**

    Label at most $2s$ vertices in W_{uv};

  else

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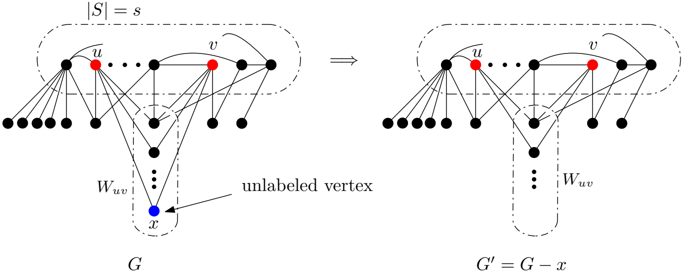

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Delete the unlabeled vertices of $V(G) \backslash S$ with at least two neighbors in S.

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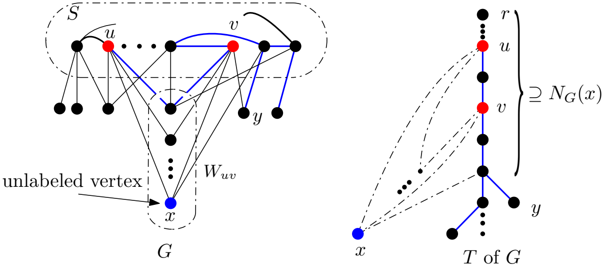

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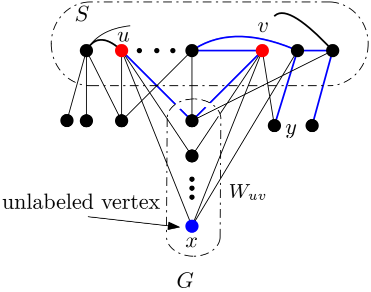

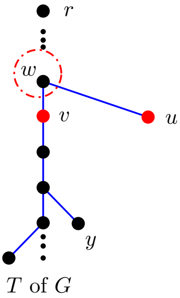

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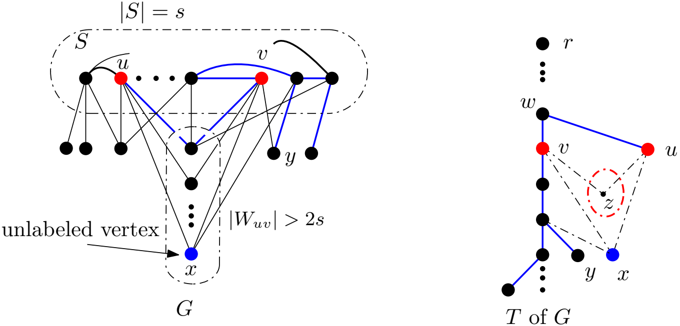

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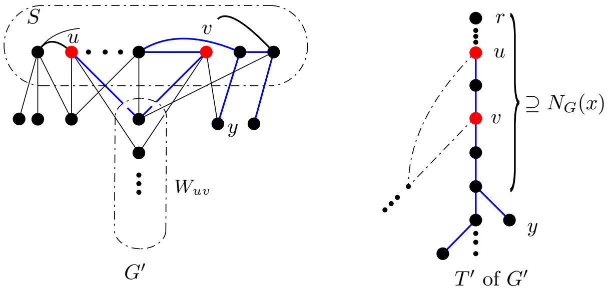

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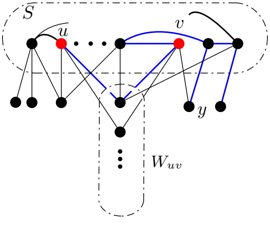
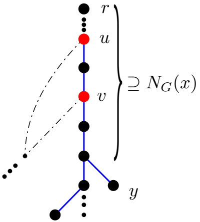


G'



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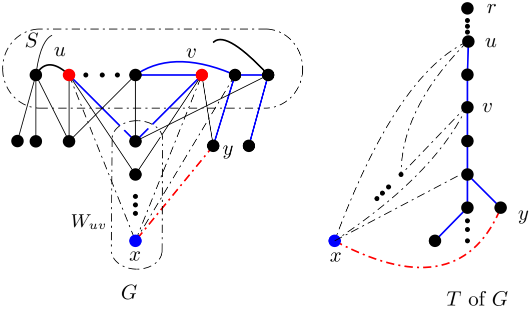

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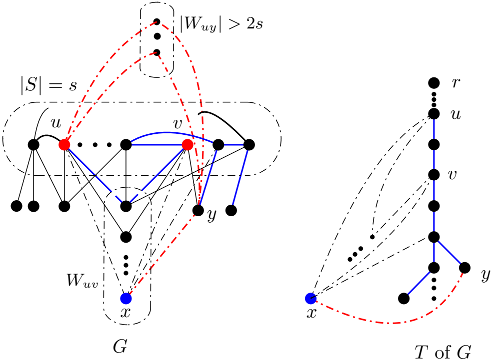

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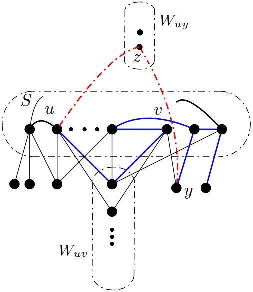

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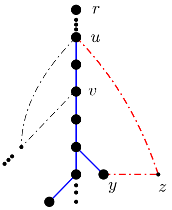

Not a DFS tree of G'

Safeness of Rule 2

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- Thus, G' has at most $s^2(s-1) + 2s + s = s^2(s-1) + 3s = \mathcal{O}(s^3)$ vertices.

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- Min-LLT and Max-LLT admit polynomial kernels with $\mathcal{O}(\tau^3)$ vertices when parameterized by the vertex cover number τ of the input graph.

Conclusion

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Thank you!