On the Parameterized Complexity of the Structure of Lineal Topologies (Depth-First Spanning Trees) of Finite Graphs: The Number of Leaves

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Introduction

For a given connected undirected graph G, a *depth-first spanning* (DFS) tree T of G is a rooted spanning tree with the property that for every edge $xy \in E(G)$ that is not an edge of T, either x is a descendant of y with respect to T, or x is an ancestor of y.

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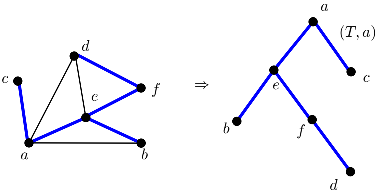

A graph G and a DFS tree (T, a) of G

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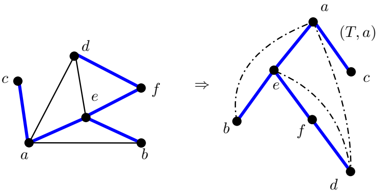
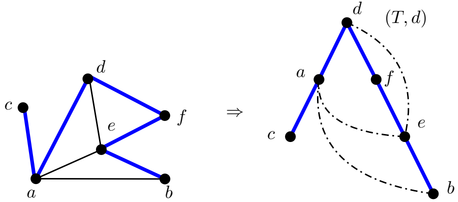


A graph G and a DFS tree (T, a) of G with back edges

- A DFS tree has also been called a *lineal spanning tree*



The spanning tree (T, d) is not a DFS tree of G.

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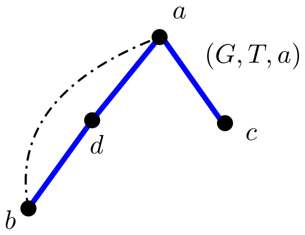
- For example, given :



$\mathcal{T} = \{\emptyset, \{ad\}, \{ac\}, \{ad, db, ba\}, \{ad, ac\}, \{ad, db, ba, ac\}\}.$

Applications

- Planarity testing and embedding [De Fraysseix, 2008; Hopcroft and Tarjan, 1974]

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- Planarity testing and embedding [De Fraysseix, 2008; Hopcroft and Tarjan, 1974]
- It can be used to define the *treedepth* of a graph [Nešetřil and de Mendez, 2012]
- DFS trees have been used to structure the search space of backtracking algorithms for solving *constraint satisfaction problems* [Freuder and Quinn, 1985]
- The problem of checking, for a given graph G and a positive integer k, whether G has a DFS tree with height at most k is NP-complete. [Fellows et al., 1988].

Problem Definitions

k-Minimum Leafy Lineal Topology (k-Min-LLT)

Input: A connected undirected graph $G = (V, E)$ and $k \in \mathbb{N}$

Question: Does G admit an LT with $\leqslant k$ leaves?

k-Maximum Leafy Lineal Topology (k-Max-LLT)

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- Related to the NP-complete MIN LEAF SPANNING TREE and MAX LEAF SPANNING TREE problems [Garey and Johnson, 1990; Lu and Ravi, 1996].

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- Related to the NP-complete Min Leaf Spanning Tree and Max Leaf Spanning Tree problems [Garey and Johnson, 1990; Lu and Ravi, 1996].
- k-Min-LLT is NP-complete.

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We consider:

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DUAL MIN LLT

Input:	A connected undirected graph $G = (V, E)$ and $k \in \mathbb{N}$
Parameter:	k
Question:	Does G admit an LT with $\leqslant n - k$ leaves?

DUAL MAX LLT

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- We show the following theorem by a *parameterized reduction* from the Multicolored Independence Set (MIS) problem.

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The k-Max-LLT *problem is W[1]-hard parameterized by* k

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*The k-*Max-LLT* problem is W[1]-hard parameterized by k and NP-complete, when considered classically*

- Dual Min-LLT and Dual Max-LLT are NP-complete when considered classically.

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- Dual Min-LLT and Dual Max-LLT are NP-complete when considered classically.

Theorem

Dual Min-LLT *and* Dual Max-LLT *are FPT parameterized by k.*

Proof:
We present a parameterized reduction from:

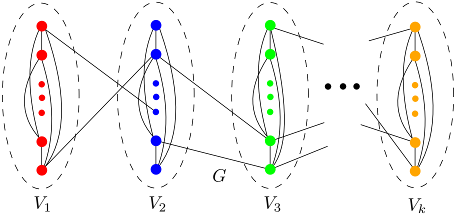
Multicolored Independent Set (MIS)

Input:	A graph $G = (V, E)$, and $f : V \to [1, k]$ with $k \in \mathbb{N}$.
Parameter:	k
Question:	Does G contain a k-colored independent set?

We assume that each color class V_i, for $i \in [1, k]$, induces a clique.

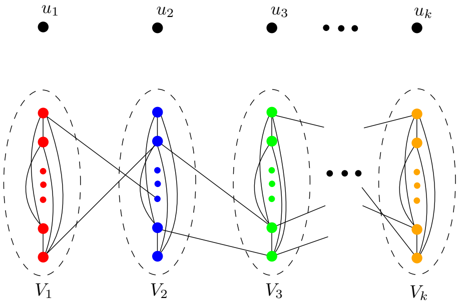
Proof:

- Given an instance (G, k) of MIS.



Proof:

- Given an instance (G, k) of MIS.
- We construct an instance (G', k') of k-Max-LLT with $k' = k$.



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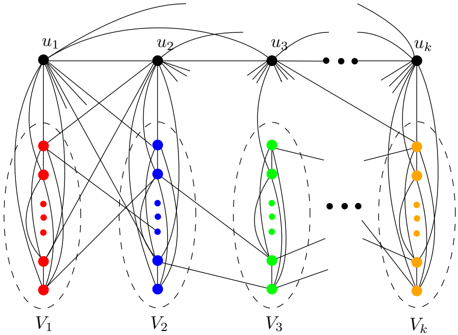

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- Given an instance (G, k) of MIS.
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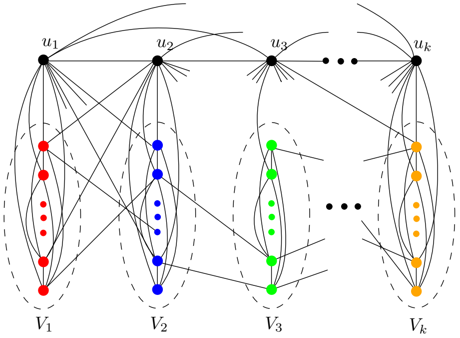
- (G', k') can be constructed in polynomial time.

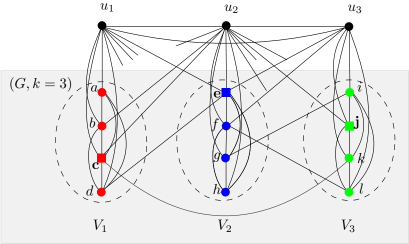


Lemma

G has a k-colored independent set $\Rightarrow$ G' admits an LT with at least k leaves.

Proof.

- Let $X = \{x_1, ..., x_k\}$ be a k-colored independent set in G.



G' and a k-colored independence set $X = \{c, e, j\}$ in G

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- Let $X = \{x_1, ..., x_k\}$ be a k-colored independent set in G.
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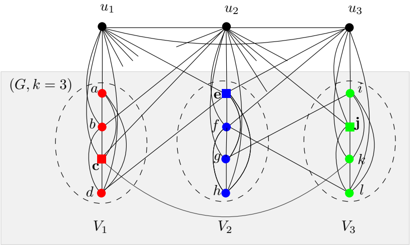

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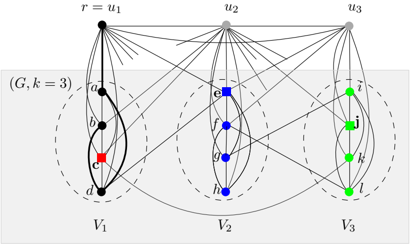

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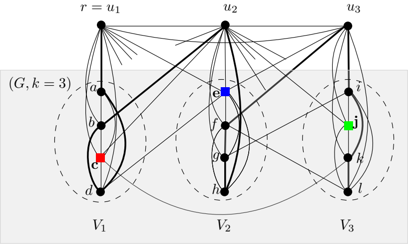

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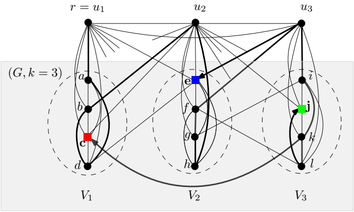

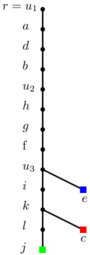

G' with $X = \{c, e, j\}$, and the resulting DFS tree T

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G has a k-colored independent set $\Leftarrow$ G' admits an LT with at least k leaves.

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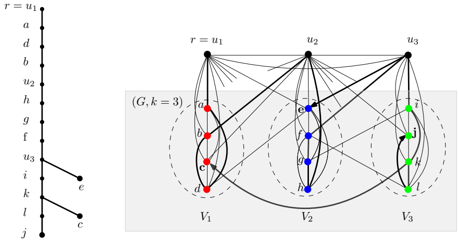
- Suppose that $k \geqslant 2$, and G' admits a DFS tree T' with at least k,



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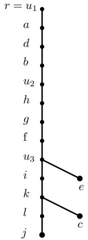

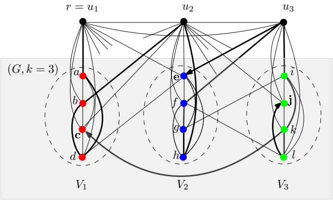

W[1]-Hardness of k-Max-LLT Parameterized by k

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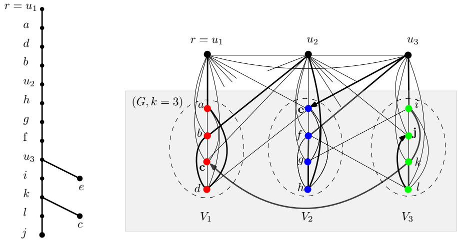
- *Claim 1: Each vertex in X belongs to at most one color class V_i in G.*



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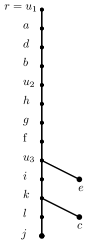
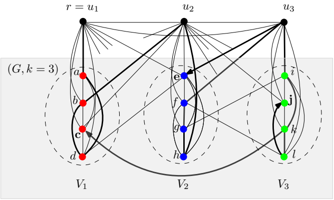
- *Claim 1: Each vertex in X belongs to at most one color class V_i in G.*
- *Claim 2: None of the vertices in X belongs to $U = \{u_1, \ldots, u_k\}$.*





To show that Dual Max-LLT and Dual Min-LLT are FPT with respect to k, we employ the following theorem:

Theorem (Courcelle, 1990)

Given $k \in \mathbb{N}$ and a fixed MSO$_2$ formula ϕ of length ℓ expressing a graph property, there is an algorithm that takes G with treewidth at most k as input and decides whether $G \models \phi$ in time $\mathcal{O}\big(f(\ell, k) \cdot n\big)$, for some computable function f.

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We express the properties of having an LT with at least $n - k$ leaves and at most $n - k$ leaves in the MSO_1 variant of *monadic second-order logic.*

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In MSO_1:

- Variables denote vertices and vertex sets
- Predicates $adj(u, v)$ and $u \in V$ are used for adjacency and membership respectively
- Quantification is allowed only over vertices and vertex sets.

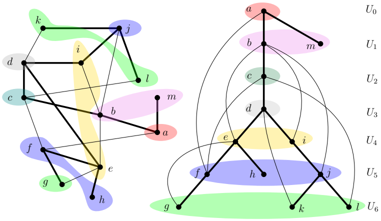
Plan:

We transform each problem to that of deciding whether the graph G has a k-sized connected vertex set X' such that:

- $G[X']$ has a DFS tree $(T_{X'}, r)$ with height $h \leqslant k$ and some additional properties that enable $(T_{X'}, r)$ to be extended to a DFS tree of G.

An LT with Bounded Height

Consider any LT (G, r, T) of height $h \in N$ in a given graph G.



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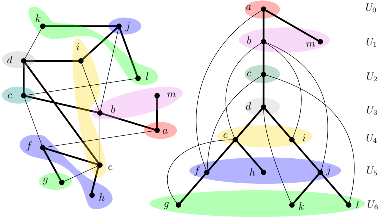
Observation:
The DFS tree (T, r) corresponds to a sequence of subsets $U_0, \ldots, U_h \subseteq V(G)$ such that:

1. *$U_0, \ldots, U_h$ is a partition of $V(G)$*
2. *U_0 contains only one element r.*



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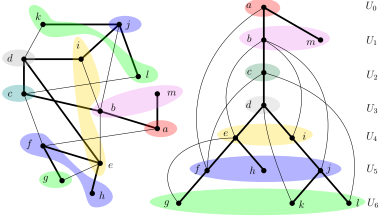

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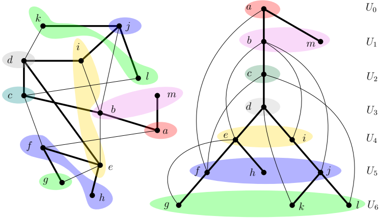
4. *For every edge uv of G, u is an ancestor of v in T.*



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4. *For every edge* uv *of* G, u *is an ancestor of* v *in* T.

Given (G, k), a sequence of subsets $(U_0, \ldots, U_h) \subseteq V(G)$ with $h \leqslant k$ is called:

- *tree-partition* if it satisfies properties (1) - (3)
- *LT-partition* if it is a *tree-partition* and satisfies property (4).

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MSO_1-formula for an LT with Bounded Height

$$\phi_h \equiv \exists_{U_0,\ldots,U_h \subseteq V} \textsc{lt-partition}(U_0, \ldots, U_h, V).$$

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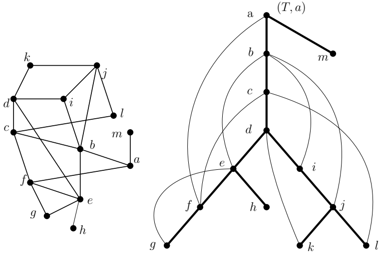
$$\begin{aligned}\textsc{tree-partition}(U_0,U_1,\ldots,U_h,V) \equiv{} & \textsc{part}(U_0,U_1,\ldots,U_h,V) \\ & \wedge \textsc{root}(U_0) \\ & \wedge \bigwedge_{i=1}^{h} (\forall_{v\in U_i}\exists_{u\in U_{i-1}}\textsc{uniq}\text{N}(v,u,U_{i-1})).\end{aligned}$$

Consider an LT (G, r, T) witnessing that (G, k) is a YES-instance.



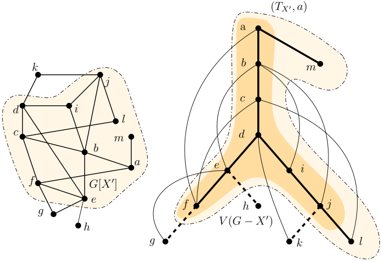
$(G, k = 10)$ with $n = 13$ and an LT (G, T, a) with 5 leaves

Characterization of Dual Max-LLT

Observation:

For any subtree $(T_{X'}, r)$ on a k-sized vertex X' set consisting of all the internal vertices and zero or more leaves of T:

1. *X' is a connected vertex cover (CVC) of G.*



$(G, k = 10)$ with $n = 13$ and a subtree $(T_{X'}, a)$ of $G[X']$

Characterization of Dual Max-LLT

Observation:
For any subtree $(T_{X'}, r)$ on a k-sized vertex X' set consisting of all the internal vertices and zero or more leaves of T:

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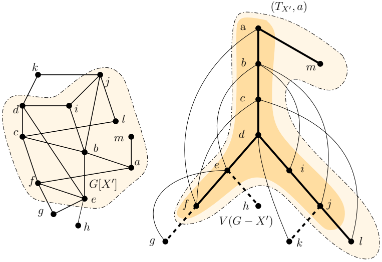

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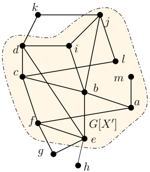
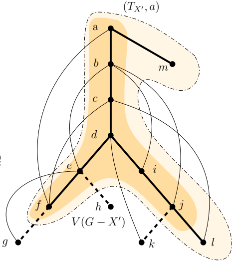
2. *$(T_{X'}, r)$ is a DFS tree of $G[X']$.*
3. *For every pair of vertices $u, v \in X'$ that have a common neighbor $y \in V(G) \backslash X'$, either u is an ancestor or a descendant of v in $(T_{X'}, r)$.*





$(G, k = 10)$ with $n = 13$ and a subtree $(T_{X'}, a)$ of $G[X']$

Characterization of Dual Max-LLT

Lemma

A graph G has an LT with at least $n - k$ leaves if and only if it has a vertex set X' with $|X'| \leqslant k$ satisfying the following properties:

1. *X' is a connected vertex cover (CVC)*
2. *$G[X']$ admits an LT partition $(U_0, ..., U_h)$ with $h \leqslant |X'|$ such that, for any vertex $y \in V(G) \backslash X'$, if y is adjacent to a pair of vertices $u, v \in X'$, then either u is the ancestor of v or v is the ancestor of u in the LT formed by $(U_0, \ldots, U_h)$.*

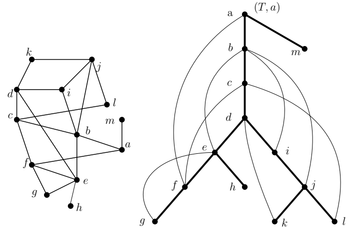
$$\psi_k \equiv \exists_{X' \subseteq V} \exists_{x_1, \ldots, x_k \in X'} \Big[\text{CONN}(X', V) \wedge \text{VC}(X', V)$$

$$\wedge \bigvee_{i \in [2,k]} \big(\exists_{U_0, \ldots, U_i \subseteq X'} \text{LT-PARTITION}(U_0, \ldots, U_i, V)$$

$$\wedge \, (\forall_{x \in V - X'} \exists_{u, v \in X'} \text{adj}(x, u) \wedge \text{adj}(x, v)) \Rightarrow (\text{ANCESTOR}(u, v, U_0, \ldots, U_i)$$

$$\vee \, \text{ANCESTOR}(v, u, U_0, \ldots, U_i)) \big) \Big].$$

Consider an LT (G, r, T) witnessing that (G, k) is a YES-instance

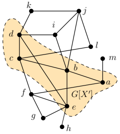


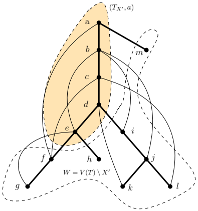
$(G, k = 5)$ with $n = 13$ and an LT (G, T, a) with 5 leaves

Observation:

Let $(T_{X'}, r)$ be any r-rooted subtree of T induced by a set of internal vertices X' of size k.

1. *$(T_{X'}, r)$ is a DFS tree of $G[X']$.*



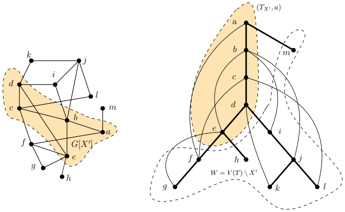


$(G, k = 5)$ with $n = 13$ and a subtree $(T_{X'}, a)$ of $G[X']$

Characterization of Dual Min-LLT

Observation:
Let $(T_{X'}, r)$ be any r-rooted subtree of T induced by a set of internal vertices X' of size k.

1. *$(T_{X'}, r)$ is a DFS tree of $G[X']$.*
2. *Every leaf of $(T_{X'}, r)$ is adjacent to a vertex in $W = V(T) \backslash X'$*



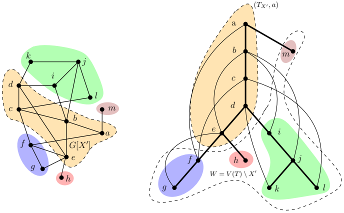
$(G, k = 5)$ with $n = 13$ and a subtree $(T_{X'}, a)$ of $G[X']$

Characterization of Dual Min-LLT

Observation:

Let $(T_{X'}, r)$ be any r-rooted subtree of T induced by a set of internal vertices X' of size k.

1. *$(T_{X'}, r)$ is a DFS tree of $G[X']$.*
2. *Every leaf of $(T_{X'}, r)$ is adjacent to a vertex in $W = V(T) \setminus X'$*
3. *For every $W' \in V(G)$ such that $G[W']$ is a maximal connected subgraph of $G - X'$, there exists some vertex $x' \in X'$ such that any vertex $x \in X'$ that has at least one neighbor in W' is an ancestor of x' in $(T_{X'}, r)$.*



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Lemma

A graph G has an LT with $\leqslant n - k$ leaves if and only if there exists a k-sized connected vertex set X' such that $G[X']$ admits an LT-partition $(U_0, \ldots, U_h)$ of height $h \leqslant k$ that satisfies properties (2) and (3).

MSO_1 Formula for Dual Min-LLT

$$\psi_k \equiv \exists_{X' \subseteq V} \exists_{W = V \setminus X'} \exists_{x_1, \ldots, x_k \in X'} \Big[\textsc{conn}(X', V)$$

$$\wedge \bigvee_{i \in [2,k]} \Big(\exists_{U_0, \ldots, U_i \subseteq X'} \textsc{lt-partition}(U_0, \ldots, U_i, X', V)$$

$$\wedge \left(\forall_{x \in X'} \textsc{leaf}(x, U_0, \ldots, U_i) \Rightarrow \exists_{y \in W} \text{adj}(x, y) \right)$$

$$\wedge \Big(\forall_{W' \subseteq W} \Big(\textsc{maxconn}(W', W, V) \Rightarrow \Big(\exists_{x' \in X'} \Big(\forall_{x_i \in X'} \exists_{w \in W'} \text{adj}(x_i, w)$$

$$\Rightarrow \left(\textsc{ancestor}(x_i, x', U_0, \ldots, U_i) \right) \Big) \Big) \Big) \Big) \Big) \Big].$$

FPT Algorithm for Dual Max-LLT

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For any given graph G and $k \in \mathbb{N}$, if G is a YES-instance of DUAL MAX-LLT*, then G admits an LT with height at most k.*

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Given a graph G and $k \in \mathbb{N}$, if G admits an LT of height at most k, then the length of any path in G is at most $2^{k+1} - 2$.

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Proof: Let T be a DFS tree of G given by DFS.

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THANK YOU